L. Szpiro, Paris: "Flatness and flatness"

0.

Let $f: \mathbb{C}^4 \rightarrow \mathbb{C}^3$ be the morphism of analytic manifolds obtained by "forgetting" a fixed coordinate

$$
\begin{array}{ccc}
  x & \leftarrow & y \\
  \uparrow & & \downarrow \\
  z & \leftarrow & t \\
\end{array}
\xrightarrow{f}
\begin{array}{ccc}
  x & \leftarrow & y \\
  \uparrow & & \downarrow \\
  z & \leftarrow & \\
\end{array}
$$

It is a flat morphism of Stein manifolds. Let $A = \mathcal{O}(\mathbb{C}^3)$ and $B = \mathcal{O}(\mathbb{C}^4)$ be the global sections of the respective sheaves of germs of analytic functions (e.g. $A$ is the ring of power series in $x, y, z$ with radius of convergence equal to infinity). One has a map, still called $f: A \rightarrow B$. The object of these pages is to prove the following which is a result of a collaboration of A. Douady and myself:

Counter example $f: A \rightarrow B$ is not a flat homomorphism of rings.

It is a sort of a surprising fact because the Stein varieties play in analytic geometry sort of the role of affine algebraic variety in algebraic geometry (see for example H.Cartan's theorems A and B and the analogues in Serre's F.A.C.).

The title is explained by the fact that even if we are able to prove that $B \otimes_A \cdot$ is not an exact functor.

The completed tensor product $B \hat{\otimes}_{A}$ (which is defined because $B$ is a nuclear space) is exact (it is one of the definition of nuclear spaces).

1. Translation in a problem of generators and relations.

Put $X = \mathbb{C}^3$ and $Y = \mathbb{C}^4$ and consider the following exact sequence of coherent $\mathcal{O}_X$-modules
\[ 0 \rightarrow \mathcal{O}_X^m \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{M} \rightarrow 0 \]

where \( m \) and \( n \) are two integers.

(no higher cohomology groups)

By the theorem one has the following exact sequence

\[ 0 \rightarrow \Gamma(X, \mathcal{O}) \rightarrow A^m \rightarrow A^n \rightarrow \Gamma(X, \mathcal{M}) \rightarrow 0 \]

If \( f: A \rightarrow B \) is flat one has the following exact sequence:

\[ 0 \rightarrow R \otimes_A B \rightarrow B^m \otimes_A \mathcal{M} \rightarrow B^n \rightarrow \mathcal{M} \otimes_A B \rightarrow 0 \]

on the other hand \( f: Y \rightarrow X \) being flat one has two exact sequences

\[ 0 \rightarrow \mathcal{O} \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y^m \rightarrow \mathcal{O}_Y^n \rightarrow \mathcal{M} \otimes \mathcal{O}_Y \rightarrow 0 \]

\[ 0 \rightarrow \Gamma(Y, \mathcal{O} \otimes \mathcal{O}_Y) \rightarrow B^m \otimes \mathcal{O}_X \rightarrow \Gamma(Y, \mathcal{M} \otimes \mathcal{O}_Y) \rightarrow 0 \]

like \( \psi = \mathcal{O} \otimes \mathcal{M} \rightarrow \Gamma(X, \mathcal{M} \otimes \mathcal{O}_Y) \) and

\[ R \otimes_A B \rightarrow \Gamma(Y, \mathcal{O} \otimes \mathcal{O}_Y) \]

(1)

Definition: Let \( A \) be a ring, \( M \) and \( N \) two \( A \)-modules \( \mathcal{M} \in M \otimes N \)

\[ \text{rank}(\mathcal{M}) = \inf \sum_{i=1}^{s} m_i \otimes n_i \quad m_i \in M, \ n_i \in N. \]

Lemma: (1) implies that the rank of an element in \( R \otimes_A B \) is bounded (by a fixed number!).

Corollary: In this situation the number of generators of \( \mathcal{O}_X \) over \( \mathcal{O}_x, \mathcal{M} \) is bounded by a number independent of \( \mathcal{M} \in \mathcal{X} \). It is a direct consequence of the theorem \( A \) which says that on a Stein manifold a coherent sheaf is generated by its global sections.

Our goal will be then obtained if we can find for any number \( k \), an ideal \( a_k \) in the ring \( \mathbb{C}[x,y,z] = \mathcal{O} \) such that
(i) \( \dim_k \mathcal{O}/\mathcal{O}_k < \infty \).
(ii) \( \mathcal{O}_k \) is generated by 4 elements.
(iii) The kernel of \( \mathcal{O}^4 \to \mathcal{O}_k \) cannot be generated by less than \( k \) elements.

Because in that case we will choose a discrete set of points \( P_1, P_2, \ldots, P_k, \ldots \)
in \( \mathbb{C}^3 = X \), and defined the coherent sheaf \( \mathcal{M} \) on \( X \) by:

a) the support of \( \mathcal{M} \) is reduced to \( P_1, P_2, \ldots, P_k, \ldots \)
b) \( \mathcal{M}_{P_k} = \mathcal{O}_{X, P_k}/\mathcal{O}_k \)

(this makes a big difference with algebraic geometry!!)

Then, the sheaf \( \mathcal{O}_X \) being coherent one has an exact sequence of coherent sheaves on \( X \)

\[
\mathcal{O} \to R \to \mathcal{O}_X^4 \to \mathcal{O}_X \to \mathcal{M} \to 0
\]

2. The proof of the existence theorem

2.1 Analysis

Let \( \mathcal{O} \) be a regular local ring of dim three, \( \mathcal{O}_t \) a primary ideal generated by four elements then one has an exact sequence

\[
\mathcal{O} \to R^t \to R^s \to R^4 \to R \to R/\mathcal{O}_t \to 0 \quad (\text{because } \text{pd}(R/\mathcal{O}_t) = 3)
\]

and then

\[
\begin{array}{c}
t + 4 = s \\
\end{array}
\]  

(2) We choose \( s \) minimal.

Lemma 2: There exists a regular \( \mathcal{O} \)-sequence \( \mathcal{f} = (f_1, f_2, f_3) \) such that \( \mathcal{f} \) can be completed on a minimal system of generators of \( \mathcal{O}_t \).

Lemma 3: With an \( \mathcal{f} \) like in Lemma 2.

Define \( \mathcal{J} \) a primary ideal in \( \mathcal{O} \) by \( \mathcal{J} \supset \mathcal{f} \) and \( \mathcal{J}/\mathcal{f} = \text{Hom}_{\mathcal{O}}(\mathcal{O}_t, \mathcal{O}_t/\mathcal{f}) \)

\[ \cong \text{Ext}^3_{\mathcal{O}}(\mathcal{O}_t, \mathcal{O}) \]

then \( \mathcal{O}/\mathcal{J} \) is a gorenstein artinian ring, and \( \mathcal{J} \) has a minimal number of generators equal between \( t \) and \( t+3 \).

Proof: \( \text{Hom}_{\mathcal{O}}(\mathcal{O}/\mathcal{J}, \mathcal{O}_t/\mathcal{f}) = \mathcal{O}_t/\mathcal{f} \)
by the assumption on $f$, $\omega_f$ is cyclic and like $\omega_f$ is a dualizing module for the artinian ring $\omega_Y$ this one is gorenstein.

\[
0 \rightarrow R \rightarrow R^4 \rightarrow R^5 \rightarrow R^7 \rightarrow \omega_f \rightarrow 0
\]

is a minimal resolution of $\omega_f$ then the end of the assertion.

**Definition:** Let $\omega$ be a ring. A **liaison** is a triple of ideals in $\omega$ ($\sigma, \mathcal{I}, f$) where

1. $f = f_1, \ldots, f_\delta$ is an $\omega$ regular sequence
2. $\omega_\sigma$ and $\omega_\mathcal{I}$ have codimension $\delta$ and $\sigma$ and $\mathcal{I}$ contains $f$
3. $\text{Hom} \omega(\omega_\sigma, \omega_f) = \omega_f$ and $\text{Hom} \omega(\omega_\mathcal{I}, \omega_f) = \omega_f$

**Remark:** To prove our existence theorem it will be enough to prove that there exist primary ideals $\mathcal{I}_k$ such that $\mathcal{I}_k$ is generated by at least $k+3$ elements and $\omega_\mathcal{I}_k$ is Gorenstein.

**2.2 "liaison" in codimension 2**

**Proposition:** Let $\omega$ be a local gorenstein ring ($\sigma, \mathcal{I}, f$) a liaison in $\omega$ such that $\sigma$ and $\mathcal{I}$ have no imbedded components then $\omega_\sigma$ is Cohen Macaulay if and only if $\omega_\mathcal{I}$ is. Moreover if $\sigma$ and $\mathcal{I}$ have no common component then $\omega_\sigma + \mathcal{I}$ is a Gorenstein ring of codimension one more than $\omega_f$, and $\sigma \cap \mathcal{I} = f$.

From the exact sequence

\[
0 \rightarrow \omega_\sigma \cap \mathcal{I} \rightarrow \omega_\sigma \oplus R/\mathcal{I} \rightarrow \omega_\sigma + \mathcal{I} \rightarrow 0
\]

one gets the exact sequence

\[
\text{Tor}_1^\omega(k, \omega_\sigma \oplus \omega_\mathcal{I}) \rightarrow \text{Tor}_1^\omega(k, \omega_\sigma + \mathcal{I}) \rightarrow \text{Tor}_2^\omega(k, \omega_f) = 0
\]

and then the number of generators of $\sigma + \mathcal{I}$ will be big if the
number of generators of \( \mathfrak{a} \) (or \( \mathcal{I} \) or of the two of them) is big.

**Proposition:** Let \( \mathfrak{a} \) be a regular local ring of dimension \( d \geq 2 \). \( \mathfrak{a} \) an ideal of \( \mathcal{O} \) of codimension 2 without imbedded components then \( \mathcal{O}/\mathfrak{a} \) is Cohen Macaulay if and only if there exists a finite number of liaisons \( l_1, l_2, \ldots, l_s \) such that \( \mathfrak{a}(l_1) = \mathfrak{a} \) and \( \mathcal{I}(l_s) \) is a regular \( \mathcal{O} \) sequence (of length \( 2d \)). Moreover the minimal number \( s \) possible is equal to the minimal number of generators of \( \mathfrak{a} \) minus 2.

**Lemma 4:** Let \( (\mathfrak{a}, \mathcal{I}, \mathfrak{f}) \) be a liaison of codimension 2 in a regular local ring \( \mathcal{O} \). Let \( V \) = the vector space spanned by \( \mathfrak{f} \) in \( \mathcal{O}/m^i \mathfrak{a} \) then one has the following correspondence:

<table>
<thead>
<tr>
<th>( \dim V )</th>
<th>( \min \text{ nb gen}(\mathfrak{a}) - \min \text{ nb gen}(\mathcal{I}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>


*J.-P. Vigué, Paris: "Differential operators on an analytic space"

**I. Definitions:**
Let \( (X, x) \) be a germ of analytic space, and let \( \mathcal{O}_{X, x} \) be the local ring of germs of analytic functions on \( (X, x) \). We get the following definition.