Outline

Introduction and Motivation

Point Isolation Problem (Chapter 2)
  Approximation Algorithm for the Point Isolation Problem
  NP-completeness of the Point Isolation Problem

Covering the Boundary of a Simple Polygon
  Approximation Factor
  Algorithm: Implementation and Running-time

Combinatorial Separation Results

Packing $\mathbb{R}^3$ with Thin Tori
Motivation: Sensor Networks

- Full Coverage: Historically the topic of interested in Sensor Networks

- Barrier Coverage: Recently, sensors used to provide *barriers* as a defense mechanism against intruders at buildings, estates, national borders etc.
Given a set of points and a set of separating objects (line segments, disks), select the minimum number of separators such that every path between two points is intersected.
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Separating Points using Disks

Problem (Point Isolation Problem [1])

Given a set $S$ of $k$ points in the plane and a collection $\mathcal{D}$ of $n$ unit disks embedded in the plane, no disk contains a point of $S$. Find a minimum cardinality subset $\mathcal{D}' \subseteq \mathcal{D}$, s.t. every path between any two points in $S$ is intersected by at least one disk in $\mathcal{D}'$.

Separating Points using Disks

Problem (Point Isolation Problem [1])

*Given a set \( S \) of \( k \) points in the plane and a collection \( \mathcal{D} \) of \( n \) unit disks embedded in the plane, no disk contains a point of \( S \). Find a minimum cardinality subset \( \mathcal{D}' \subseteq \mathcal{D} \), s.t. every path between any two points in \( S \) is intersected by at least one disk in \( \mathcal{D}' \).*

Motivation

Sensor Networks:

- *Full coverage*: expensive
- *Barrier coverage*: only detect certain spacial transitions among the observed objects

Motivated by this:

- [1] presents a constant-factor approximation algorithm
- But problem complexity open.

Problem (Circle 2-Cells-Separation Problem)

Given a set of $n$ Circle and two points $s$ and $t$, select the minimum number of circles one needs to retain so that any $s$-$t$ path intersects some of the retained segments.

Theorem (Cabello et al., SoCG13)

Can be computed in time $O(nk + n^2 \log n)$, $k$ being the number of Circle intersections.
A recursive approximation Algorithm for \( k \)-points

\[
\text{recSep}(Q, D)
\]

1. If \(|Q| \leq 1\), return \(\emptyset\).
2. For every pair of points \( s, t \in Q \), invoke the subroutine to find a minimum cardinality subset \( B_{s,t} \subseteq D \) such that \( B_{s,t} \) separates \( s \) and \( t \).
3. Let \( B_Q \) denote a minimum size subset \( B_{s,t} \) over all pairs \( s, t \in Q \).
4. Let \( Q_1 \) and \( Q_2 \) be the partition of \( Q \) into two subsets such that each subset corresponds to points in the same face induced by the arrangement of disks in \( B_Q \).

\[
\text{return } B_Q \cup \text{recSep}(Q_1, D) \cup \text{recSep}(Q_2, D)
\]
A recursive approximation Algorithm for $k$-points
A recursive approximation Algorithm for $k$-points
A recursive approximation Algorithm for $k$-points
A recursive approximation Algorithm for $k$-points
A recursive approximation Algorithm for $k$-points
The union complexity of \( m \) disks is \( 2(3m - 6) \).
Approximation factor = 6

1. Optimal separator for 2 points has exactly one bounded face.
2. Associate the arrangement $B$ computed in a recursive call with the point $s$ in the bounded face, i.e. $B_s = B$.
3. Let $F_s$ be the disks in $OPT$ surrounding $s$. ($|B_s| \leq |F_s|$)
4. $$\sum_{p \in S} |B_p| \leq \sum_{p \in S} |F_p|.$$ 
5. No two faces $F_s, F_t$ of $OPT$ have any arcs in common $\Rightarrow$.
6. Decomposing $OPT$ into $F_{p_1}, \ldots, F_{p_k}$ yields a partition of the boundary arcs $B(OPT)$
7. $$|\text{recSep}(S, D)| \leq \sum_{p \in S} |B_p| \leq \sum_{p \in S} |F_p| = |B(OPT)| \leq 6|OPT|$$
Main Result

Theorem

*The Point Isolation Problem is NP-complete if the number of points is not fixed.*
Problem (Planar unweighted Multiterminal Cut Problem)

Given a planar unweighted graph $G = (V, E)$ and a set $S \subseteq V$ of $k$ terminals, find a minimum cardinality set $E' \subseteq E$ such that in $G' = (V, E \setminus E')$ there is no path between any two nodes in $S$. 
Relevant Problems

Problem (Planar unweighted Multiterminal Cut Problem)

Given a planar unweighted graph $G = (V, E)$ and a set $S \subseteq V$ of $k$ terminals, find a minimum cardinality set $E' \subseteq E$ such that in $G' = (V, E \setminus E')$ there is no path between any two nodes in $S$.

Theorem (Johnson, Papadimitriou, Seymour, Yannakakis, 1994)

NP-complete if $k$ is not fixed.
Planar Subdivision Problem

Problem (Planar Subdivision Problem)

Given a planar graph $G = (V, E)$ embedded in the plane and a set $S$ of $k$ points properly contained in the faces of $G$ with no face containing more than one point, find the minimum cardinality set $E' \subseteq E$ such that in the embedding of the reduced graph $G' = (V, E')$, no two points of $S$ are contained in the same face.
Planar Subdivision Problem

Problem (Planar Subdivision Problem)

Given a planar graph $G = (V, E)$ embedded in the plane and a set $S$ of $k$ points properly contained in the faces of $G$ with no face containing more than one point, find the minimum cardinality set $E' \subseteq E$ such that in the embedding of the reduced graph $G' = (V, E')$, no two points of $S$ are contained in the same face.
Complexity of the Planar Subdivision Problem

Proposition

*The Planar Subdivision Problem is NP-complete if \( k \) is not fixed, even for connected graphs.*
Given an planar unweighted Multiterminal Cut Instance
Construct the geometric dual multigraph
subdivide each edge
put a point in each relevant face
solve the planar subdivision problem
retrieve solution to the Multiterminal Cut problem
retrieve solution to the Multiterminal Cut problem
Goal is:

**Theorem**

*The Point Isolation Problem is NP-complete if the number of points is not fixed.*
Proof outline for Point Isolation Problem

1. Given a connected instance \( I = (G, S) \) of the Planar Subdivision Problem
2. Build equivalent straight line embedding on an \( n \times n \) grid [2]
3. Replace each edge by a path of \( c_E \) many unit disks
4. Replace each vertex by a cycle of \( c_V \) many unit disks.
5. For each \( s \in S \) put point into the corresponding face in the disk arrangement.
6. Solve the Point Isolation Instance \( f(I) \) and retrieve solution for \( I \)

Edge Gadget

- Every edge gadget consists of a path of $c_E$ many unit disks

- $c_E$, $a$, $h$ and $s$ are constant for all edge gadgets

- $1 - (2s + 2a) \leq b \leq \sqrt{2n} - (2s + 2a)$, depending on the length of the embedded edge $e$. 
Vertex Gadget

\( c_V \) many disks of radius \( r \) arranged on a circle of radius \( s \) around vertex \( v \), with \( c_V = \lceil \pi s / r \rceil \).

- Incident edge gadgets intersect some disks in the edge gadget.
- Removing a single disks from an edge gadget merges two regions.
Dimension Constraints

Assure that no edge gadget intersect any non-incident vertex gadget.

Observation

In an \( n \times n \) grid, the minimum distance between any line \( l \) through two grid points and any grid point not on \( l \) is \( 1/\sqrt{2n^2 - 2n + 1} \).
Dimension Constraints

Assure that no two incident edge gadget intersect each other.

Observation

In an $n \times n$ grid, for any grid point $p$ the minimum angle between any two distinct lines, each going through $p$ and at least one other grid point, is larger than $2 \arctan \frac{1}{(6n^2)}$. 
Dimension Constraints

\[
\begin{align*}
2(r + s) &< 1 & \text{VGs disjoint} \\
2(r + s + h/2) &< (2n^2 - 2n + 1)^{-\frac{1}{2}} & \text{Non-inc. VGs, EGs disj.} \\
2\frac{h}{2} &< (2n^2 - 2n + 1)^{-\frac{1}{2}} & \text{Non-incident EGs disj.} \\
\left\lceil \frac{\sqrt{2n-2s}}{2r} \right\rceil - 2\left\lceil \frac{a}{2r} \right\rceil &< \left\lceil \frac{h}{2r} \right\rceil \cdot \left\lfloor \frac{1-2(a-s)}{2r} - 1 \right\rfloor & \forall \text{EGs contain } c_E \text{ disks}
\end{align*}
\]

Satisfied with \( r = \frac{1}{40n^4} \) and \( h = \frac{1}{12n^2} \) for all \( n \geq 2 \), when setting \( a = 1/4 \) and \( s = 1/(7n^2) \).
Equivalence

Lemma

I of the Planar Subdivision Problem has a solution of size \( \leq B \)
\iff
\( f(I) \) of the Point Isolation Problem has a solution of size
\( \leq C_E(B + 1) - 1 \)

Note that \( nc_V < c_E \)
Containment in NP

Given $\mathcal{D}' \subseteq \mathcal{D}$, to an instance $I = (S, \mathcal{D})$ of the Point Isolation Problem and $B \in \mathbb{N}$:
Verify in polynomial time if $\mathcal{D}'$ is a solution of $I$ with $|\mathcal{D}'| \leq B$.

- Build the embedded intersection graph $G$ of $\mathcal{D}'$, add vertex at each edge crossing.
- this gives a line segment arrangement
- $S$ separated in $\mathcal{D}' \iff S$ separated in arrangement
- Point location in arrangement for each point reject $\iff$ the same face of is reported twice.
Complexity of Multiterminal Cut on Unit Disk Graphs

Theorem

The Multiterminal Cut Problem remains NP-complete on unit disk graphs if $k$ is not fixed.
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Problem Setting

Problem

*Given a region, bounded by a piecewise linear closed border, such as a fence, place few guards inside the fenced region, such that wherever an intruder cuts through the fence, the closest guard is at most a distance one away.*
Problem Setting

Problem

Given a region, bounded by a piecewise linear closed border, such as a fence, place few guards inside the fenced region, such that wherever an intruder cuts through the fence, the closest guard is at most a distance one away.
Geodesic Unit Disk

Definition
A *geodesic unit disk* centered at a point $v$ in a polygon $P$ is the set of all points in $P$ whose shortest path distance to $v$ is at most 1.

Problem (Boundary Coverage)
*Given a simple polygon, cover its boundary with the minimum number of Geodesic Unit Disks.*
Greedily cover the longest uncovered boundary portion
⇒ $\Omega(\log n)$ approximation
Contiguously cover the longest uncovered boundary portion $\Rightarrow \geq 2$ approximation
Contiguously cover the longest uncovered boundary portion $\Rightarrow \geq 2$ approximation
ContiguousGreedy

c ← v₁, v_u ← v₂
S ← ∅

while ∂P not covered:
  1. If c⁻¹v_u is longer than 2
      compute centers on c⁻¹v_u at steps of 2; add them to S; update c
  2. Update v_u to the first vertex s.t. ∂P[c, v_u] cannot be covered
     by a single disk, using Exponential and Binary Search with
     predicate TestCover
  3. Use AugmentShort to cover the vertices between c and v_u,
     and a maximal portion of the edge v_u⁻¹v_u; add new center to
     S and update c

end while

return S
Figure: Illustration of the $\delta$-thin polygon where an $\epsilon$-approximate contiguous extension algorithm results in an approximation ratio larger than 2.
Definition
A coloring of $\partial P$ is a function $\gamma : \partial P \rightarrow \mathbb{N}$. The number of colors used by $\gamma$ is defined as the cardinality of the image of $\gamma$.

Definition
A block is a connected component of $\partial P$ colored with a single color.

Definition
We let $\partial P_i$ denote the subset of the polygon boundary colored with color $i$ and we call each connected component of $\partial P \setminus \partial P_i$ a pocket of $\partial P$ induced by color $i$ (see Fig. ??(b)).
Approximation Factor

Definition
A coloring of $\partial P$ is called \textit{crossing-free} if for any two distinct colors $i, j$, it holds that $\partial P_j$ is contained in a single pocket induced by color $i$.

Definition
For a collection $\mathcal{D} = \{D_1, \ldots, D_k\}$ of disks covering $\partial P$, a \textit{disk-coloring} of $\partial P$ w.r.t. $\mathcal{D}$ is a function $\gamma_{\mathcal{D}} : \partial P \rightarrow \{1, \ldots, k\}$, such that $\gamma(x) = i \Rightarrow x \in D_i$, i.e., a point on $\partial P$ can only be colored with one of the indices of the disks covering it.
Definition
For a coloring $\gamma$, two of its colors $r$ and $b$ cross each other, if there are two pockets induced by color $r$ containing blocks of color $b$. 
Crossing Colors

Lemma

In any disk-coloring, if two colors \( r \) and \( b \) cross each other, one of the following holds: 1) There exists a pocket induced by color \( r \) which contains blocks of color \( b \) and all these blocks can be re-colored with color \( r \), s.t. the resulting coloring is still a disk-coloring. 2) There exists a pocket induced by color \( b \) which contains blocks of color \( r \) and all these blocks can be re-colored with color \( b \), s.t. the resulting coloring is still a disk-coloring.
Two Lemmas for the Approximation Ratio

Lemma

For any collection of disks covering $\partial P$, there exists a crossing free disk-coloring of $\partial P$. 
Approximation Ratio

Lemma

For a crossing-free coloring $\gamma$ using $\kappa$ colors, let $\Pi_\gamma$ be the set of blocks induced by $\gamma$. If $\kappa > 1$ then $|\Pi_\gamma| \leq 2(\kappa - 1)$. 
Approximation Ratio

Theorem
The number of disk centers placed by \texttt{ContiguousGreedy} is at most $2|\text{OPT}| - 1$.

Proof.
If $|\text{OPT}| = 1$ then, by its greedy nature, \texttt{ContiguousGreedy} also uses only one disk. If $|\text{OPT}| > 1$, let $\gamma_{\text{OPT}}$ be a crossing free disk-coloring of $\partial P$ w.r.t. $\text{OPT}$, whose existence is guaranteed by Lemma 23. We let $(B_1, B_2, ..., B_m)$ be the collection of blocks induced by $\gamma_{\text{OPT}}$ ordered as they appear on $\partial P$ in clockwise order, with $B_1$ the block containing $v_1$. We split $B_1$ at $v_1$ into two blocks $B_l$ and $B_r$, with $B_l$ being the portion of $B_1$ counterclockwise from $v_1$, and $B_r = B_1 \setminus B_l$.

Now observe that by the greedy nature, every disk $D$ computed by \texttt{ContiguousGreedy} extends $\Gamma$ so that $\Gamma \cup D$ fully covers at least one new block in the sequence $(B_r, B_2, ..., B_m, B_l)$. Therefore, after computing at most $m + 1$ disks, $\Gamma = \partial P$. By Lemma 24, it holds that $m \leq 2(|\text{OPT}| - 1)$ and the theorem follows.
Running time: $O(n \log^2 n)$

- Let $\Gamma$ be the currently contiguously covered portion of $\partial P$.
- We need to place a new disk to extend $\Gamma$ maximally.
- How (fast) can we do this?
Place the new disk

- Let $c$ be the endpoint of $\Gamma$
- We need to find the first vertex $v_u$ of $P$ which can not be covered with a disk containing $c, v_i, v_{i+1}, \ldots, v_{u-1}$.
- Use exponential and binary search w.r.t. interval covered.
- But how to do a single test on $c, v_i, v_{i+1}, \ldots, v_k$?
Intermezzo: Furthest Site Geodesic Voronoi Diagram. Aronov et. al ’88

- combinatorial complexity $O(n + k)$
- constructed in time $O((n + k) \log(n + k))$
- geodesic center of sites obtained for free, root of bisector tree
Single Test: Is $c, v_i, v_{i+1}, \ldots, v_k$ covered?

- Compute shortest path $\pi(c, v_k)$ from $c$ to $v_k$
- Construct weakly simple polygon $Q = (c, v_i, v_{i+1}, \ldots, v_k) \bullet \pi(c, v_k)$
- $Q$ is geodesic convex, i.e. for any $u, v \in Q : \pi(u, v) \in Q$
- In $Q$ Compute the Furthest site Geodesic Voronoi Diagram of $c, v_i, v_{i+1}, \ldots, v_k$
- This gives their Geodesic Center
- Check if all sites are within distance 1 $\Rightarrow c$ to $v_k$ coverable by single disk.
- Time: $O(|Q| \log n)$
Lemma

Let $Q$ be the set of polygons constructed during the algorithm. It then holds that $\sum_{Q \in Q} |Q| = O(n \log n)$.

Thus if we only spend $O(|Q| \log n)$ time per such polygon, total running time is $O(n \log^2 n)$. 
Given $v_u$, place disk to maximize $\Gamma'$ in time $\text{Time: } O(|Q| \log n)$

- look at intersection of unit disks centered at $c, v_i, v_{i+1}, \ldots, v_{u-1}$ $A$ (but don’t compute it)
Given \( v_u \), place disk to maximize \( \Gamma' \) in time \( \text{Time}: O(|Q| \log n) \)

- look at intersection of unit disks centered at 
  \( c, v_i, v_{i+1}, \ldots, v_{u-1} \) \( A \) (but don’t compute it)

**Observation**

Let \( \overline{\alpha \beta} \) be a line-segment, such that for all \( a \in A \), \( d(\alpha, a) \leq 1 \) and \( d(\beta, a) > 1 \). For a point \( c \in \overline{\alpha \beta} \) and any disk center \( q \) furthest away from \( c \), it holds that \( d(c, A) = 1 \) if and only if

1. \( d(c, q) = 2 \) and \( \pi(c, q) \cap \partial D(q) \in A \) or
2. \( d(c, I) = 1 \),

with \( I \) denoting the disk-disk intersection points on \( \partial A \).
Hardness

Theorem

*Boundary Coverage is NP-hard in polygons with holes.*
Proof Outline

Theorem (GJS77)

*Vertex Cover on planar graphs of maximum degree 3 is NP-hard.*

Observation

*In any graph, replacing an edge by a path of odd length $k$ increases a vertex cover by $(k - 1)/2.*
Proof Outline

- Taken a planar max deg 3 instance $G = (V, E)$
- Embed a $G' = (V', E')$, obtained from $G$ by replacing edges $e$ by paths of odd length $l_e$, consisting of straight line edges of length 1.
- Edge: two paths of length 1 connecting $u$ and $v$. At 0.5, an additional path of length 0.5 is attached.
- $G$ has an $VC$ of size $M \iff P(G')$ has a covering of size $\sum_{e \in E}(l_e - 1)/2 + M$
Covering Long Perimeter Polygons (Preliminary Result)

Theorem

If the polygon perimeter $L$ is at least $n^{1+\delta}$, with $\delta > 0$, a simple linear time algorithm achieves an approximation ratio which goes to one as $L/n$ goes to infinity.

\[
\frac{|D|}{|OPT|} \leq 1 + \frac{O(n)}{|OPT|} = 1 + \frac{O(n)}{\Omega(L)} = 1 + O\left(\frac{n}{L}\right) = 1 + o(1).
\]
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Basic Definitions

Definition
For a set $S$ of $n$ points in the plane, a linear bipartition $P$ of $S$ is a set \{\( U, S \setminus U \)\} consisting of two disjoint nonempty subsets of $S$ which respectively are fully contained in the two open half-planes bounded by some line.

Example
\[ S = \{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq \mathbb{R}^2 \]
\[ P = \{\{v_1, v_2, v_3, v_6\}, \{v_4, v_5\}\} \]
Separating Family

Definition
A set $\mathcal{P}$ of linear bipartitions is called a linear separating family for $S$ if for every distinct $p, q \in S$ there is a $P = \{U, S \setminus U\}$ in $\mathcal{P}$, s.t. $p \in U$ and $q \in S \setminus U$.
We will refer to linear separating families as separating families.

Example

\[
\begin{align*}
\{ \{v_1, v_2, v_3, v_4\}, \{v_5, v_6\}\} \\
\{ \{v_1, v_2, v_3, v_6\}, \{v_4, v_5\}\} \\
\{ \{v_1, v_3, v_4, v_5, v_6\}, \{v_2\}\} \\
\{ \{v_1, v_4, v_5, v_6\}, \{v_2, v_3\}\}
\end{align*}
\]
Definition
A separating family $\mathcal{P}$ for $S$ is called \textit{minimal}, if no proper subset of $\mathcal{P}$ is a separating family for $S$.

Example

\begin{align*}
S &= \{v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq \mathbb{R}^2 \\
&= \{\{v_1, v_2, v_3, v_4\}, \{v_5, v_6\}\} \\
&\quad \cup \{\{v_1, v_2, v_3, v_6\}, \{v_4, v_5\}\} \\
&\quad \cup \{\{v_1, v_3, v_4, v_5, v_6\}, \{v_2\}\} \\
&\quad \cup \{\{v_1, v_4, v_5, v_6\}, \{v_2, v_3\}\} \\
&\quad \cup \{\{v_1, v_2, v_5, v_6\}, \{v_3, v_4\}\}
\end{align*}
The Problem we Investigated:

“Given \( n \) convex point in the plane, how many...?”

Example

How many minimal separating families of size 3 exist for 6 points in convex position?
Results

For $n$ points in convex position in the plane we present:

- A Bijection to some restricted class of Edge Covers.

- Enumeration results for minimal separating families of size:
  - $\lceil n/2 \rceil$, i.e. the *minimum* size case,
  - $n - 1$, i.e. the *maximum* size case,
Idea

Separating family → abstract graph on \([n]\)

Which graphs correspond to minimal separating families?
Necessary condition: Every vertex needs an incident edge.
Definition
An edge cover on \([n]\) is a set \(H \subseteq \binom{[n]}{2}\) of edges such that every vertex in \([n]\) is incident to at least one edge in \(H\).

Proposition
OEIS[A054548]: The number \(E(n, m)\) of edge covers of size \(m\) on \([n]\) is

\[
E(n, m) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{i}{m}.
\]
Observation

Not every edge cover corresponds to a separating family.

Observation

We need some crossings on the edge cover.
Definition
Two edges \( \{a, b\}, \{c, d\} \in \binom{[n]}{2} \) are called crossing, if \( a < c < b < d \).

Definition
Two connected components \( C, D \) are crossing if there are crossing edges \( e, f \) with \( e \in C \) and \( f \in D \).
Crossingly Connected Edge Cover

Definition
An edge cover $H$ is called *crossingly connected* if any two edges $e, f$ in $H$ are contained in the same connected component or there is a sequence $C_1, \ldots, C_k$ of crossing components s.t. $e \in C_1$ and $f \in C_k$. 

![Diagram of a graph with nodes and edges illustrating the concept of crossingly connected edge cover.]
First Bijection

Proposition

There is a bijection between the set of all separating families for $n$ points in convex position in the plane and the set of all crossingly connected edge covers on $[n]$. 
Enumerating Crossingly Connected Edge Covers

Proposition

For $C(n, m)$ being the number of all crossingly connected edge covers of size $m$ on $[n]$ the following equation holds:

$$E(n, m) = \sum_{i,j>0} C(i,j) \sum_{k_1+\cdots+k_i=n-i, l_1+\cdots+l_i=m-j} E(k_1, l_1) \cdots E(k_i, l_i).$$
Bijection for *minimal* separating families

**Theorem**

*There is a bijection between the set of all minimal separating families for n points in convex position in the plane and the set of all crossingly connected edge covers on \([n]\) such that there is no crossingly connected cycle containing a path of length \(\geq 3\).*

![Diagram of a set of points and connecting edges demonstrating the bijection]
Corollary

Any edge cover of size $k$ on $[n]$ which satisfies the condition in the last Theorem consists of $n - k$ non-crossing trees, each of size at least 2.

Thus:

- A family of size $n - 1$ corresponds to a non-crossing tree.
- A family of size $\lceil n/2 \rceil$ corresponds to:
  - A crossingly connected perfect matching on $[n]$, if $n$ is even.
  - A crossingly connected almost perfect matching on $[n]$, if $n$ is odd.
It is well known: [P. Hilton and J. Pedersen, 1991], [S. Dulucq, J. G. Penaud, 1993]

**Theorem**

The number of non-crossing trees on $n$ points on a circle is

$$\frac{1}{2n-1} \binom{3n-3}{n-1}.$$  

Thus we conclude:

**Theorem**

The number of minimal separating families of size $n - 1$ for a convex $n$-point set in the plane is

$$\frac{1}{2n-1} \binom{3n-3}{n-1}.$$
Enumeration Results for *minimum size* Families

- **n even:**
  crossingly connected perfect matchings on \([n]\)

- **n odd:**
  crossingly connected almost perfect matchings on \([n]\)
Definition
Let \( e_n \) denote the number of crossingly connected perfect matchings on \([2n]\).  
[M. Klazar, 2003]

Theorem
The generating function 
\[
E = \sum_{n \geq 1} e_n x^n = x + x^2 + 4x^3 + 27x^4 + \cdots ,
\]
satisfies the differential equation 
\[
E' = \frac{E^2 + E - x}{2xE}
\]
n odd

Definition
Let $f_n$ denote the number of crossingly connected almost perfect matchings on $[2n + 1]$.

Theorem
The generating function

$$F = \sum_{n \geq 1} f_n x^n = 3x + 15x^2 + 126x^3 + 1395x^4 + \cdots,$$

is related to $E$ by

$$xF = E^3 E' + 2xE^3 E'' + E^2 - x^2$$
The Main Theorem

**Theorem**

The number $s_n$ of minimal linear separating families of minimum size for $n$ points in convex position in the plane is

$$s_n = \begin{cases} (k - 1) \sum_{i=1}^{k-1} s_{2i}s_{2(k-i)} & \text{if } n = 2k \\ s_{n+1} \frac{(2k+1)(k+1)}{2k} & \text{if } n = 2k + 1 \end{cases}$$

for all $n \geq 3$, with $s_2 = 1$. 
Asymptotic Behavior

Denoting by \( q_k \) the number of near-matchings on \([2k + 1]\) the following proposition holds.

Proposition

\[
\lim_{k \to \infty} \frac{f_k}{q_k} = e^{-1}
\]

It holds that

\[
\lim_{k \to \infty} e_{k+1} \frac{2^{k+1}(k+1)!}{(2k+2)!} = e^{-1}
\]

and it is easy to show that \( q_k = \frac{(2k + 1)!}{(2^k(k - 1)!)} \).

Thus it follows that

\[
\lim_{k \to \infty} \frac{f_k}{q_k} = \lim_{k \to \infty} e_{k+1} \frac{2^{k+1}(k+1)!}{(2k+2)!} \frac{(2k + 2)(2k + 1)}{4k^2} = e^{-1}.
\]
Basic Definitions

Definition
We mean by a *bipartition* a set partition consisting of at most two components. We call a bipartition *proper* if it consists of exactly two components.

Definition
A family \( \mathcal{P} \) of bipartitions of a set \( S \) is called a *separating* family for \( S \) if every two elements of \( S \) can be cut by some bipartition in \( \mathcal{P} \). A separating family \( \mathcal{P} \) for \( S \) is *minimal* if no proper subfamily of \( \mathcal{P} \) is a separating family for \( S \).
Example

Let \( S = \{1, 2, 3, 4\} \). Let \( P_1, P_2, Q_1, Q_2, Q_3 \) be the bipartitions defined as:

\[
P_1 = \{\{1, 2\}, \{3, 4\}\}, \quad Q_1 = \{\{1\}, \{2, 3, 4\}\},
\]
\[
P_2 = \{\{1, 3\}, \{2, 4\}\}, \quad Q_2 = \{\{1, 2\}, \{3, 4\}\},
\]
\[
P_3 = \{\{1, 2, 3\}, \{4\}\}.
\]

Then the family of bipartitions \( \{P_1, P_2\} \) is a minimal separating family of minimum size for \( S \), while \( \{Q_1, Q_2, Q_3\} \) is that of maximum size.
Outline

Introduction and Motivation

Point Isolation Problem (Chapter 2)
  Approximation Algorithm for the Point Isolation Problem
  NP-completeness of the Point Isolation Problem

Covering the Boundary of a Simple Polygon
  Approximation Factor
  Algorithm: Implementation and Running-time

Combinatorial Separation Results

Packing $\mathbb{R}^3$ with Thin Tori
Packing $\mathbb{R}^3$ with Thin Tori

- Sphere: only non-tiling body for we know the exact packing density [Hales’05]
- Very limited amount of literature studying packings involving non-convex objects:
- We like to extend this line of research by considering packings with the possibly simplest non-convex shape, the torus.
Packing $\mathbb{R}^3$ with Thin Tori

Problem

*Can $\mathbb{R}^3$ be packed at a positive density with tori of major radius $R = 1$ and minor radius $r$ going to 0?*

Volume of torus $= \Theta(r^2)$
Volume of bounding box of torus $= \Theta(r)$

$\Rightarrow$

Volume of torus / Volume of bounding box of torus $\rightarrow 0$
Proof Strategy: Lattice Arrangement of thick tori

- With major radius 1 and minor radius $r$, (t.b.d.)
- Center those thick tori at lattice points
- Then pack each of those thick tori with thin tori of major radius 1 and minor radius going to 0
Packing density $\delta_L(r)$ for thick tori

All thick tori are centered at lattice points generated by vectors:

\[
\begin{pmatrix}
2 + \sqrt{3}r \\
0 \\
r
\end{pmatrix}, \quad \begin{pmatrix}
1 + r \\
\sqrt{3}(1 + r) \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
2r
\end{pmatrix}
\]

Packing density of thick tori w.r.t. $\mathbb{R}^3$ as function of $r$:

\[
\delta_L(r) = \frac{\text{volume of single thick torus}}{\text{volume of lattice parallelepiped}} = \frac{\pi^2 r}{\sqrt{3} (2 + \sqrt{3}r)(1 + r)}
\]
Next show that a single thick torus can be packed with thin tori (with minor \(\rightarrow 0\)) at density \(\delta_T(r) > 0\).

Thus overall packing density is

\[ \delta(r) = \delta_L(r)\delta_T(r) > 0 \]

But how to pack a single thick torus?
Idea for Packing: Villarceau Circles

Definition
A pair of Villarceau circles on the surface of the torus is produced by cutting a torus diagonally through the center with a bitangential plane.

Observation
These circles have a radius which correspond to the major radius of the torus.
Villarceau Circles
Villarceau Circles
Villarceau Circles
Proof Strategy: Replace circle with a thin torus $T(1, s)$

Problem
Given a torus $T(R, r)$, lying in the $xy$-plane and two Villarceau circles $c_0, c_1$ lying in the bitangential planes $H_0, H_1$ respectively. How much must the minimum angular distance $\alpha$ around the $z$-axis between $H_0$ and $H_1$ be such that the minimum distance between $c_0$ and $c_1$ is at least $2s$?

The $2s$ neighborhood of the surface of a thick torus can be packed with $2\pi/\alpha$ many thin tori.
Proof Strategy: Nested Construction

Place \( \left\lfloor \frac{r-s}{2s} \right\rfloor \) many nested constructions into the thick \((1, r)\) torus.
Pairwise linking of tori

Problem
How many tori of major radius 1 and minor radius \( s \) can be pairwise linked, as a function of \( s \)?

Fact: The Villarceau circles of one torus are all pairwise linked.

Easy Claim: The Villarceau circles of the nested torus sequence are all pairwise linked.