Self-referentiality in Constructive Semantics of Intuitionistic and Modal Logics

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Introduction

Self-referentiality

Future Works

BHK Semantics for Intuitionistic Logic

Justification Logics JL

Realization

Introduction
BHK Semantics, Proposed

- Brouwer: Only constructions achieve mathematical objects, and only proofs achieve mathematical truth.
- BHK Stipulation (propositional clauses):
  - \( \bot \) has no proof;
  - a proof of \( \phi \land \psi \) is the pair of a proof of \( \phi \) and a proof of \( \psi \);
  - a proof of \( \phi \lor \psi \) is a proof of \( \phi \) or a proof of \( \psi \);
  - a proof of \( \phi \rightarrow \psi \) is a construction that returns a proof of \( \psi \) whenever a proof of \( \phi \) is given.
- Provability as Truth.

| Yu, Junhua | Self-ref. in Constructive Seman. of Int. & Modal Logics |
Frameworks related to BHK

- Gödel: A calculus virtually S4, no provability semantics offered;
- Kleene's interpretation: (codes) of programs, undecidable;
- Curry – Howard isomorphism: \( \lambda \)-terms that encode intuitionistic ND proofs;
- Kreisel's attempt: inconsistent, then restricted.
Intuitionistic Propositional Logic IPC

- Primitive connectives: \( \bot, \land, \lor, \text{ and } \rightarrow. \)
- Axioms:

\[
\begin{align*}
\bot & \rightarrow \phi \\
\phi \rightarrow \psi & \rightarrow \phi \\
(\phi \rightarrow \psi & \rightarrow \chi) \rightarrow (\phi \rightarrow \psi) \rightarrow \phi \rightarrow \chi \\
\phi \land \psi & \rightarrow \phi \\
\phi \land \psi & \rightarrow \psi \\
\phi & \rightarrow \phi \lor \psi \\
\psi & \rightarrow \phi \lor \psi \\
(\phi & \rightarrow \chi) \rightarrow (\psi \rightarrow \chi) \rightarrow \phi \lor \psi \rightarrow \chi.
\end{align*}
\]

- Rule: MP.
- Glivenko’s Theorem:
  - IPC \( \vdash \neg \neg \alpha \) iff CPC \( \vdash \alpha. \)
BHK Semantics for Intuitionistic Logic
Justification Logics JL
Realization

BHK Semantics Formalized

IPC $\hookrightarrow$ S4 $\hookrightarrow$ LP $\hookrightarrow$ Proofs

- **IPC $\leftrightarrow$ S4: Gödel embedding**
  - Embedding $(\cdot)^\Delta$: □ every subformula; □ means provability.
  - IPC $\vdash \phi$ iff S4 $\vdash \phi^\Delta$.

- **S4 $\leftrightarrow$ LP: Artemov’s realization**
  - Realization $(\cdot)^r$: Substitute □’s by terms.
  - S4 $\vdash \phi$ iff LP $\vdash \phi^r$ for some normal realization $\phi^r$.

- **LP $\leftrightarrow$ Proofs: Artemov’s completeness theorem**
  - Arithmetical interpretation.
  - LP($CS$) $\vdash \phi$ iff $\phi$ is provably valid under $CS$. 
Justification Logics J, JD, JT, J4, LP (Artemov et.al.)

\[ \phi := \bot \mid p \mid \phi \rightarrow \phi \mid t : \phi, \]
where \( t := !^n c \mid x \mid t \cdot t \mid t + t \) (for J, JD, JT)
and \( t := c \mid x \mid t \cdot t \mid t + t \mid ! t \) (for J4, LP).

<table>
<thead>
<tr>
<th>Name</th>
<th>Form (scheme)</th>
<th>Adopted in</th>
</tr>
</thead>
<tbody>
<tr>
<td>A0</td>
<td>Classical propositional axioms</td>
<td>All</td>
</tr>
<tr>
<td>A1.1</td>
<td>( t : \phi \rightarrow \phi )</td>
<td>JT, LP</td>
</tr>
<tr>
<td>A1.2</td>
<td>( t : \bot \rightarrow \bot )</td>
<td>JD</td>
</tr>
<tr>
<td>A2</td>
<td>( t_1 : (\phi \rightarrow \psi) \rightarrow (t_2 : \phi \rightarrow t_1 \cdot t_2 : \psi) )</td>
<td>All</td>
</tr>
<tr>
<td>A3</td>
<td>( t : \phi \rightarrow ! t : t : \phi )</td>
<td>J4, LP</td>
</tr>
<tr>
<td>A4.1</td>
<td>( t_1 : \phi \rightarrow t_1 + t_2 : \phi )</td>
<td>All</td>
</tr>
<tr>
<td>A4.2</td>
<td>( t_2 : \phi \rightarrow t_1 + t_2 : \phi )</td>
<td>All</td>
</tr>
<tr>
<td>MP</td>
<td>( \alpha \rightarrow \beta, \alpha \vdash \beta )</td>
<td>All</td>
</tr>
<tr>
<td>AN.1</td>
<td>( \vdash c : A )</td>
<td>J4, LP</td>
</tr>
<tr>
<td>AN.2</td>
<td>( \vdash !^n c : !^{n-1} c : \cdots ! c : c : A )</td>
<td>J, JD, JT</td>
</tr>
</tbody>
</table>
Definition:
- J4, LP: A set of formulas of the form $c : A$;
- J, JD, JT: A set of formulas of the form $!^n c : 1^{n-1} c : \cdots : c : A$, and downward closure is required;

The collection of $AN$ rule-applications in a proof automatically form a constant specification, named the one called for by that proof.
Realization

- **Realizer**
  - A mapping: the language of modal logics ML $\mapsto$ that of a JL;
  - Assigns a term to each $\Box$-occurrence in the input formula;

- **Realization**
  - A potential realization of a modal formula $\phi$ is the mapping image $\phi^r$ of that formula under some realizer $(\cdot)^r$;
  - A realization is a potential one that is a JL-theorem;

- **Realization theorem (Artemov et.al.)**
  - For any modal formula $\phi$:
    - Let $X$ range from $\{K, D, T, K4, S4\}$,
    - and $Y$ range from $\{J, JD, JT, J4, LP\}$, resp.,
  - Then what follows are equivalent:
    - $X \vdash \phi$;
    - $Y \vdash \phi^r$ for some normal realization $r$. 
Self-referentiality
Self-referentiality in Justification Logics
Self-referential JL-formulas

(Recalled) Justification language

\[ \phi ::= \bot \mid p \mid \phi \rightarrow \phi \mid t : \phi; \]
\[ t ::= c \mid x \mid t \cdot t \mid t + t \mid !t \text{ (for J4, LP)}, \]
\[ t ::=!^nc \mid x \mid t \cdot t \mid t + t \text{ (for J, JD, JT)}. \]

\[ t : \phi(t) \]
\[ t \text{ is interpreted as} \]
- a proof of an assertion about \( t \) itself;
- a justification of an assertion about \( t \) itself;
- a set of formulas containing \( \phi(t) \).

Even possible \( c : A(c) \).
Self-referentiality of $CS$

- Directly self-referential if $c : A(c) \in CS$;
- Self-referential if $\{c_1 : A_1(c_2), \ldots, c_n : A_n(c_1)\} \subseteq CS$;
- Let $CS^{\not\exists_1} := \{c : A | c \text{ does not occur in } A\}$
  - The largest non-directly self-referential constant specification.
- Let $CS^{\not\exists} := \{c_i : A | c_j \text{ does not occur in } A \text{ for } i \leq j\}$
  - The largest non-self-referential constant specification module renaming.
Self-referentiality in Modal Logics
Let $X \in \{K, D, T, K4, S4\}$ and $Y \in \{J, JD, JT, J4, LP\}$, resp.;

- $X^\Downarrow_1 := \{X \vdash \phi \mid Y(\mathcal{CS}^\Downarrow_1) \vdash \phi^r \text{ for some realizer } (\cdot)^r\}$.
- $X^\Downarrow := \{X \vdash \phi \mid Y(\mathcal{CS}^\Downarrow) \vdash \phi^r \text{ for some realizer } (\cdot)^r\}$.

Existing results (all by Kuznets):

- $K^\Downarrow = K$ (even normally);
- $D^\Downarrow = D$ (even normally);
  - Proof-theoretically.
- $T^\Downarrow_1 \not\subseteq T$, instanced by $\neg\Box\neg(p \rightarrow \Box p)$;
- $K4^\Downarrow_1 \not\subseteq K4$, instanced by $\Box\neg(p \rightarrow \Box p) \rightarrow \Box \bot$;
- $S4^\Downarrow_1 \not\subseteq S4$, instanced by $\neg\Box\neg(p \rightarrow \Box p)$;
  - M/F-model-theoretically.
Non-self-referential Fragment of ML (continued)

Let $X \in \{K, D, T, K4, S4\}$ and $Y \in \{J, JD, JT, J4, LP\}$, resp.;
- $X^{\varphi_1} := \{X \vdash \phi \mid Y(CS^{\varphi_1}) \vdash \phi^r \text{ for some realizer } (\cdot)^r\}$.
- $X^{\varphi} := \{X \vdash \phi \mid Y(CS^{\varphi}) \vdash \phi^r \text{ for some realizer } (\cdot)^r\}$.

Our results: assume $X \in \{T, K4, S4\}$
- $X^{\varphi_1}$ is closed under necessitation;
- $X^{\varphi_1}$ contains all axioms of $X$;
- $X^{\varphi_1}$ is not closed under MP.
  - prehistoric graphs + Kuznets’ theorems.
- $T^{\varphi_1} \cup K4^{\varphi_1} \subsetneq S4^{\varphi_1}$;
- $S4 \setminus S4^{\varphi_1} \not\subseteq T$, instance: $\Diamond\Box(\Diamond\Box p \rightarrow \Box p)$;
  - $S4 \setminus S4^{\varphi_1} \not\subseteq K4$ by Kuznets.
- $[T^{\varphi}, S4^{\varphi}] \cap K4 \not\subseteq K4^{\varphi_1}$, instance: $\Box p \rightarrow \Diamond\Box\Diamond p$. 
The System \textbf{G3[st4]}

\begin{align*}
\text{Ax.} & \quad \frac{\rho, \Gamma \Rightarrow \Delta, \rho}{\Gamma \Rightarrow \Delta, \phi} \\
L \rightarrow. & \quad \frac{\Gamma \Rightarrow \Delta, \phi \quad \psi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \psi, \Gamma \Rightarrow \Delta} \\
L \Box. & \quad \frac{\phi, \Box \phi, \Gamma \Rightarrow \Delta}{\Box \phi, \Gamma \Rightarrow \Delta} \\
4 \Box. & \quad \frac{\Theta, \Box \Theta \Rightarrow \eta}{\Box \Theta, \Gamma \Rightarrow \Delta, \Box \eta}
\end{align*}

\begin{align*}
\text{L} \perp. & \quad \frac{\bot, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta, \psi} \\
R \rightarrow. & \quad \frac{\Gamma \Rightarrow \Delta, \phi \rightarrow \psi}{\Box \Theta \Rightarrow \eta} \\
R \Box. & \quad \frac{\Box \Theta, \Gamma \Rightarrow \Delta, \Box \eta}{\Theta \Rightarrow \eta} \\
K \Box. & \quad \frac{\Box \Theta, \Gamma \Rightarrow \Delta, \Box \eta}{\Box \Theta, \Gamma \Rightarrow \Delta, \Box \eta}
\end{align*}

- \textbf{G3cp}: Ax, L \perp, L \rightarrow, R \rightarrow;
- \textbf{G3s}: G3cp with L \Box, R \Box;
- \textbf{G3t}: G3cp with L \Box, K \Box;
- \textbf{G34}: G3cp with 4 \Box.
Prehistoric Graph and Prehistoric Loop

- Prehistoric graph \( \mathcal{P}(\mathcal{T}) := (F, \prec_L, \prec_R, \prec) \),
  - \( F \): the set of positive families in the proof tree \( \mathcal{T} \),
  - \( \prec = \prec_L \cup \prec_R \), and
  - e.g., in

\[
([RK4] \Box). \frac{\Sigma(\Box h_1) \Rightarrow \eta(\Box h_2)}{\Box \Theta, \Gamma \Rightarrow \Delta, \Box i \eta}
\]

we have \( h_1 \prec_L i \) and \( h_2 \prec_R i \).

- A prehistoric loop is a loop in \( \mathcal{P}(\mathcal{T}) \).
Let $X \in \{T, K4, S4\}$, and $Z \in \{G3t, G34, G3s\}$, resp.;

$$X \otimes := \{ \phi \mid \text{sequent } \Rightarrow \phi \text{ has a loop-free proof in } Z \}.$$  

Our results:

- $X \otimes$ is decidable;
- $X \otimes$ is closed under necessitation;
- $X \otimes$ is closed under substitution;
- $X \otimes$ contains all axioms of $X$;
- $X \otimes$ is not closed under MP;
- $X \otimes \subseteq X \emptyset$. 

Illustration:

Known fact: if S4 \not\vdash \sigma, then \diamond \Box \sigma \notin X^\otimes.

\[ \Rightarrow \neg \Box \neg \Box \sigma \quad (R\neg) \]
\[ \Box \neg \Box \sigma \Rightarrow \quad (L\Box) \]
\[ \neg \Box \sigma, \Box \neg \Box \sigma \Rightarrow \quad (L\neg) \]
\[ \Box \neg \Box \sigma \Rightarrow \Box \sigma \quad (R\Box) \]
\[ \Box \neg \Box \sigma \Rightarrow \quad \sigma \]

Loop of length 1.
Self-referentiality in the BHK Semantics
Basic Embedding

Basic potential embedding $(\cdot)^\times$: prop. form. to modal form.

$$
\begin{align*}
    p_+^\times &\equiv \Box^+ p, & p_-^\times &\equiv \Box^- p, \\
    \perp_+^\times &\equiv \Box i+ \perp, & \perp_-^\times &\equiv \Box i- \perp,
\end{align*}
$$

$$(\phi \odot \psi)^\times_+ \equiv \Box^+ i\odot (\Box^k \odot \phi_+^\times \odot \Box^l \odot \psi_+^\times) \text{ for } \odot \in \{\land, \lor\},
$$

$$(\phi \odot \psi)^\times_- \equiv \Box^- i\odot (\Box^k \odot \phi_-^\times \odot \Box^l \odot \psi_-^\times) \text{ for } \odot \in \{\land, \lor\},
$$

$$(\phi \rightarrow \psi)^\times_+ \equiv \Box^+ i\rightarrow (\Box^k \rightarrow \phi_-^\times \rightarrow \Box^l \rightarrow \psi_+^\times),
$$

$$(\phi \rightarrow \psi)^\times_- \equiv \Box^- i\rightarrow (\Box^k \rightarrow \phi_+^\times \rightarrow \Box^l \rightarrow \psi_-^\times), \text{ and}
$$

$$
\phi^\times \equiv \phi_+^\times.
$$

- Is unpolarized, if it satisfies $\phi_+^\times \equiv \phi_-^\times$.
- Is a basic embedding if being faithful.
  - Known embeddings with only $\perp, \rightarrow, \Box$ as primitives;
  - A bit over-general here, possible applications elsewhere.
Let $X \in \{\text{IPC}, \text{IPC} \rightarrow\}$:

- $X \emptyset_1(\times) := \{X \vdash \phi \mid \phi^\times \in S4 \emptyset_1\}$,
- $X \emptyset(\times) := \{X \vdash \phi \mid \phi^\times \in S4 \emptyset\}$,
- $X \otimes(\times) := \{X \vdash \phi \mid \phi^\times \in S4 \otimes\}$.

Our results:

- $\text{IPC} \emptyset_1 \subsetneq \text{IPC}$, instanced by $\neg\neg (\text{CPC} \ \setminus \ \text{IPC})$,
- $\text{IPC} \emptyset_1 \subsetneq \text{IPC} \rightarrow$, instanced by $p \rightarrow q \rightarrow p \rightarrow p \rightarrow q \rightarrow q$;
  - M-model theoretically.
- $X \otimes(\times) \subseteq X \emptyset(\times) \subseteq X \emptyset_1(\times)$.

Assume $(\cdot)^\times$ unpolarized for the rest:

- $\text{IPC} \otimes(\times)$ includes all axioms of $\text{IPC} \rightarrow$,
- $\text{IPC} \otimes(\times), \text{IPC} \emptyset(\times), \text{IPC} \emptyset_1(\times)$ each is not closed under MP;
- $\text{IPC} \otimes(\times)$ includes all axioms of $\text{IPC} \rightarrow$,
- $\text{IPC} \otimes(\times), \text{IPC} \emptyset(\times), \text{IPC} \emptyset_1(\times)$ each is not closed under MP.
  - By conservativity.
Future Works
Suggested Future Works

- Self-referentiality and impredicativity;
- Applications of prehistoric graph in proof theory;
- Basic embedding applied to other propositional-language/modal-language logic pairs;
- Visser’s BPL;
  - length of loops?
- Criterion for self-referentiality;
  - decidable?
  - limits of M/F-models?
- Direct self-referentiality v.s. self-referentiality;
  - length of loops?
- Any theorem self-referential in T but not in S4?
Thanks!